A new approach to bounds on mixing

Flavien Léger

Courant Institute of Mathematical Science
NYU

Outline

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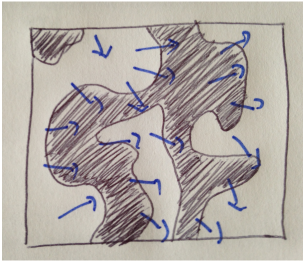
Setting

Incompressible passive scalar advection

$$\begin{cases} \partial_t \theta + \operatorname{div}(u\theta) = 0 \\ \operatorname{div}(u) = 0 \\ \theta(0, \cdot) = \theta_0 \end{cases}$$



Mixing

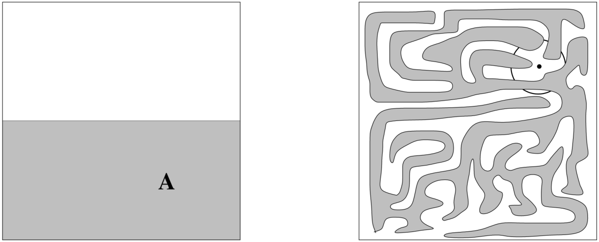
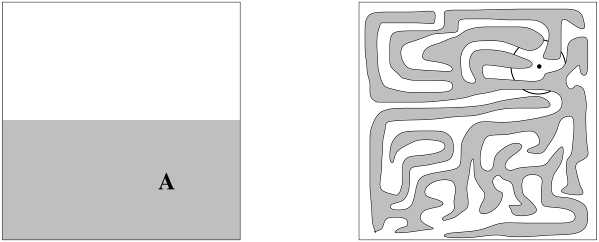


Figure credits: A. Bressan, 2003

Mixing



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Natural question: how well can we mix?

Mixing

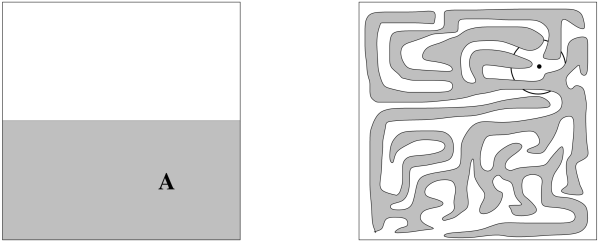


Figure credits: A. Bressan, 2003

Natural question: how well can we mix?

- ▶ Need an energetic constraint on the flow.
  Also need to quantify mixing.

Our problem

How much can an incompressible flow mix, under energetic constraint?

Energetic constraint

Cost of stirring $\theta(0,\cdot)$ to $\theta(T,\cdot)$ is

$$\int_0^T \|\nabla u(t,\cdot)\|_{L^p}\, dt$$

For $p = 2$ it is the energy transferred to a Stokes flow

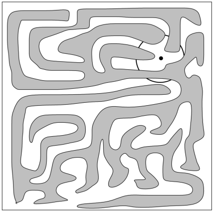
$$\begin{cases} -\Delta u = f - \nabla p \\ \operatorname{div} u = 0 \end{cases}$$

since: $\int f \cdot u\, dx = \int |\nabla u|^2\, dx$

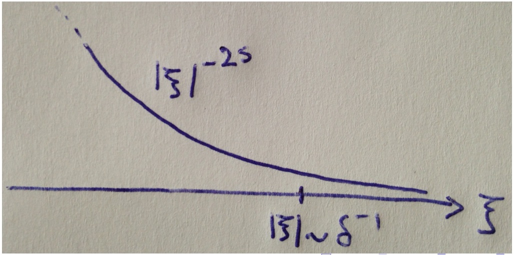
Measuring mixing

Definition (Mixing measure)

$$\varepsilon(t) := \|\theta(t,\cdot)\|^2_{\dot{H}^{-s}} = \int |\xi|^{-2s} \, |\hat{\theta}(t,\xi)|^2 \, d\xi$$



$s = 1$: $\dfrac{\|\theta(t,\cdot)\|_{\dot{H}^{-1}}}{\|\theta(t,\cdot)\|_{L^2}}$ scales as length.



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Bressan's conjecture

Conjecture (Bressan, 2006)

Let $\varepsilon(t)$ = mixing measure of $\theta(t,\cdot)$. Then

$$\varepsilon(t) \geq C^{-1} \exp\left(-C \int_0^t \left\|\nabla u(t',\cdot)\right\|_1 dt'\right)$$

Bressan's conjecture

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L^1: not known
L^p: solved ($p > 1$) (Crippa – De Lellis, 2008)

Previous results

- Crippa – De Lellis (2008)
  - Binary mixtures, Lagrangian coord
  - log of derivative in physical space
  - Use of maximal function
- Seis (2013)
  - Binary mixtures
  - Optimal transportation distance
  - Also use of maximal function

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Our approach

Consider

$$\mathcal{V}(\theta) = \int_{\mathbf{R}^d} \log|\xi|\, |\hat{\theta}(\xi)|^2 \, d\xi$$

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Then

$$\mathcal{V}\big(\theta(t,\cdot)\big) - \mathcal{V}\big(\theta_0\big) \leq C \, \|\theta_0\|_\infty \, \|\theta_0\|_{p'} \int_0^t \big\|\nabla u(t',\cdot)\big\|_p \, dt'$$

$(p > 1)$

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$(p > 1)$
Implies a restriction on mixing

$$\|\theta(t,\cdot)\|_{\dot{H}^{-1}} \ge C^{-1} \exp\left(-C \int_0^t \left\|\nabla u(t',\cdot)\right\|_p \, dt'\right)$$

Features

- Control of the log of the derivative
- Stronger than $\dot{H}^{-1}$ norm
- Technique is different

Decay of mixing measures

Simple convexity inequality: if $\|\theta_0\|_{L^2} = 1$,

$$\|\theta(t,\cdot)\|_{\dot{H}^{-1}} \geq \exp\left(-\mathcal{V}(\theta(t,\cdot))\right)$$

- Mixing measure decay at most exponentially.

Recall main theorem:

$$\mathcal{V}\big(\theta(t,\cdot)\big) - \mathcal{V}\big(\theta_0\big) \leq C \left\|\theta_0\right\|_\infty \left\|\theta_0\right\|_{p'} \int_0^t \left\|\nabla u(t',\cdot)\right\|_p \, dt'$$

Stronger than $\dot{H}^{-1}$

Compare

$$\int_0^T \|\nabla u(t,\cdot)\|_p \, dt \leq M \implies \int |\xi|^{-2} |\hat{\theta}(T,\xi)|^2 \, d\xi \gtrsim \exp(-M)$$

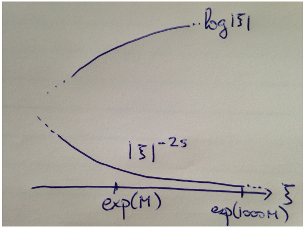
Stronger than $\dot{H}^{-1}$

Compare

$$\int_0^T \|\nabla u(t,\cdot)\|_p \, dt \leq M \implies \int |\xi|^{-2} \, |\hat{\theta}(T,\xi)|^2 \, d\xi \gtrsim \exp(-M)$$

and

$$\int_0^T \|\nabla u(t,\cdot)\|_p \, dt \leq M \implies \int \log|\xi| \, |\hat{\theta}(T,\xi)|^2 \, d\xi \lesssim M$$



Corollary: on the blowup of Sobolev norms

If $\|\nabla u(t,\cdot)\|_{L^2} \leq C$:

- $\displaystyle\int |\hat{\theta}(t,\xi)|^2 \, d\xi = C$
- $\displaystyle\int \log|\xi| |\hat{\theta}(t,\xi)|^2 \, d\xi \leq C\,(1+t)$
- $\displaystyle\int (\log|\xi|)^2 |\hat{\theta}(t,\xi)|^2 \, d\xi \leq C\,(1+t)^2$
- $\displaystyle\int |\xi|^{2s} \, |\hat{\theta}(t,\xi)|^2 \, d\xi$ can blow up, for any $s > 0$

Outline

Outline of proof

1. Write $\mathcal{V}(f) = \int \log|\xi|\,|\hat{f}(\xi)|^2\,d\xi$ in physical space. Morally

$$\mathcal{V}(f) = \iint \frac{f(x)\,f(y)}{|x-y|^d}\,dx\,dy$$

2. Time derivative is

$$\frac{d}{dt}\mathcal{V}(\theta) = \iint \theta(x)\,\theta(y)\,\frac{u(x)-u(y)}{|x-y|}\cdot\frac{x-y}{|x-y|^{d+1}}\,dx\,dy$$

3. Incompressibility implies

$$\frac{d}{dt}\mathcal{V}(\theta) = \iint \theta(x)\,\theta(y)\,(m_{xy}\nabla u) : K(x-y)\,dx\,dy$$

4. Hölder-type bounds. Mixing result follows by elementary arguments.

More details

1. Recall

$$\mathcal{V}(f) = \int_{\mathbf{R}^d} \log|\xi|\,|\hat{f}(\xi)|^2\,d\xi$$

In physical space:

$$\mathcal{V}(f) = \alpha_d \left(\frac{1}{2} \iint_{|x-y|\le 1} \frac{|f(x)-f(y)|^2}{|x-y|^d}\,dx\,dy - \iint_{|x-y|>1} \frac{f(x)f(y)}{|x-y|^d}\,dx\,dy \right) + \beta_d \|f\|_{L^2}^2$$

2. Time-derivative of $\mathcal{V}$ along the flow $\partial_t\theta + \operatorname{div}(u\theta) = 0$:

$$\frac{d}{dt}\mathcal{V}(\theta(t,\cdot)) = c_d \text{ PV} \iint \theta(t,x)\theta(t,y)\frac{u(t,x)-u(t,y)}{|x-y|}\cdot\frac{x-y}{|x-y|^{d+1}}\,dx\,dy$$

3. Incompressibility constraint $\operatorname{div} u = 0 \to$ multilinear singular integral can be written as

$$\frac{u(t,x)-u(t,y)}{|x-y|}\cdot\frac{x-y}{|x-y|^{d+1}} = (m_{xy}\nabla u(t,\cdot)) : K(x-y)$$

with K a Calderón–Zygmund kernel.

End of proof

4. Multilinear singular integral bounds (Seeger, Smart, Street):

$$\mathrm{PV} \iint \theta(t,x)\theta(t,y)\big(m_{xy}\nabla u(t,\cdot)\big) : K(x-y)\, dx\, dy \leq C \left\|\theta(t,\cdot)\right\|_{\infty} \left\|\theta(t,\cdot)\right\|_{p'} \left\|\nabla u(t,\cdot)\right\|_{p}$$

Hence

$$\left| \frac{d}{dt} \mathcal{V}\big(\theta(t,\cdot)\big) \right| \leq C \left\|\theta_0\right\|_{\infty} \left\|\theta_0\right\|_{p'} \left\|\nabla u(t,\cdot)\right\|_{p}$$

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A word on the harmonic analysis

Trilinear form

$$\Lambda(a,\theta,\phi) = \mathrm{PV} \iint K(x-y)\,(m_{xy}a)\,\theta(x)\,\phi(y)\,dx\,dy$$

$m_{xy}a$ = average of a on $[x,y]$

- Christ-Journé (1987): $C(a,\theta,\phi) \lesssim \|\theta\|_q \|\phi\|_{q'} \|a\|_\infty$
- Seeger, Smart, Street (2015): $\|a\|_p$ $(p>1)$

Outline

- Summary
  New approach to

$$\varepsilon(t) \geq C^{-1} \exp\left(-C \int_0^t \left\|\nabla u(t')\right\|_p \, dt'\right)$$

  $(p > 1)$. Use of

$$\mathcal{V}(\theta(t,\cdot)) = \int \log|\xi| \, |\hat{\theta}(t,\xi)|^2 \, d\xi$$

- Still open
  Can we get Bressan's conjecture: unlikely.
  Easier version than Bressan? $L \log L$ ok.
  Note: no L^1 result available yet.

Thank you for your attention!