A new approach to bounds on mixing

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A new approach to bounds on mixing

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Outline

Introduction

- Mixing
- Bressan's conjecture

Main result

- New perspective
- Proof
- A remark



Summary

Perspective

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Mixing

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Perspective

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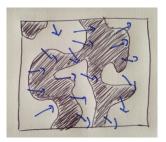
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Setting

Incompressible passive scalar advection

$$\begin{cases} \partial_t \theta + \operatorname{div}(u\theta) = 0\\ \operatorname{div}(u) = 0\\ \theta(0, \cdot) = \theta_0 \end{cases}$$



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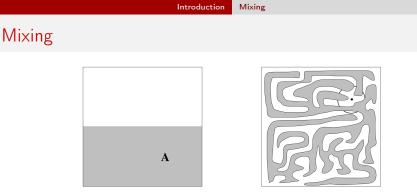


Figure credits: A. Bressan, 2003

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Figure credits: A. Bressan, 2003

Natural question: how well can we mix?

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Figure credits: A. Bressan, 2003

Natural question: how well can we mix?

► Need an energetic constraint on the flow. Also need to quantify mixing.

Our problem

How much can an incompressible flow mix, under energetic constraint?

Mixing

Energetic constraint

Cost of stirring $\theta(0, \cdot)$ to $\theta(T, \cdot)$ is

$$\int_0^T \|\nabla u(t,\cdot)\|_{L^p} dt$$

For p = 2 it is the energy transferred to a Stokes flow

$$\begin{cases} -\Delta u = f - \nabla p \\ \operatorname{div} u = 0 \end{cases}$$

since: $\int f \cdot u \, dx = \int |\nabla u|^2 \, dx$

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Mixing

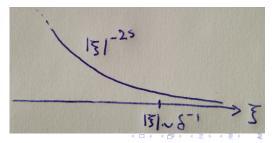
Measuring mixing

Definition (Mixing measure)

$$arepsilon(t) := \| heta(t,\cdot)\|^2_{\dot{H}^{-s}} = \int |\xi|^{-2s} \, |\hat{ heta}(t,\xi)|^2 \, d\xi$$



$$s=1: rac{\| heta(t,\cdot)\|_{\dot{H}^{-1}}}{\| heta(t,\cdot)\|_{L^2}}$$
 scales as length.



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Bressan's conjecture

Conjecture (Bressan, 2006)

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Let $\varepsilon(t) = \text{mixing measure of } \theta(t, \cdot)$. Then

$$\varepsilon(t) \geq C^{-1} \exp\left(-C \int_0^t \left\| \nabla u(t', \cdot) \right\|_1 dt'\right)$$

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Bressan's conjecture

Conjecture (Bressan, 2006)

Let $\varepsilon(t) = \text{mixing measure of } \theta(t, \cdot)$. Then

$$\varepsilon(t) \geq C^{-1} \exp\left(-C \int_0^t \left\| \nabla u(t', \cdot) \right\|_1 dt'\right)$$

 L^1 : not known L^p : solved (p > 1) (Crippa – De Lellis, 2008)

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Previous results

- Crippa De Lellis (2008)
 - Binary mixtures, Lagrangian coord
 - log of derivative in physical space
 - Use of maximal function
- Seis (2013)

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- Binary mixtures
- Optimal transportation distance
- Also use of maximal function

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Our approach

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Consider

$$\mathcal{V}(heta) = \int_{\mathsf{R}^d} \log |\xi| \, |\hat{ heta}(\xi)|^2 \, d\xi$$

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Our approach

Consider

$$\mathcal{V}(heta) = \int_{\mathbf{R}^d} \log |\xi| \, |\hat{ heta}(\xi)|^2 \, d\xi$$

Then

$$\mathcal{V}(heta(t,\cdot)) - \mathcal{V}(heta_0) \leq C \left\| heta_0
ight\|_{\infty} \left\| heta_0
ight\|_{p'} \int_0^t \left\|
abla u(t',\cdot)
ight\|_p dt'$$

(p > 1)

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Our approach

Consider

$$\mathcal{V}(heta) = \int_{\mathbf{R}^d} \log |\xi| \, |\hat{ heta}(\xi)|^2 \, d\xi$$

Then

$$\mathcal{V}(\theta(t,\cdot)) - \mathcal{V}(\theta_0) \leq C \|\theta_0\|_{\infty} \|\theta_0\|_{p'} \int_0^t \|\nabla u(t',\cdot)\|_p dt'$$

(p > 1)Implies a restriction on mixing

$$\|\theta(t,\cdot)\|_{\dot{H}^{-1}} \geq C^{-1} \exp\left(-C \int_0^t \left\|\nabla u(t',\cdot)\right\|_p \, dt'\right)$$

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Features

- Control of the log of the derivative
- Stronger than \dot{H}^{-1} norm
- Technique is different

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Decay of mixing measures

Simple convexity inequality: if $\|\theta_0\|_{L^2} = 1$,

$$\| heta(t,\cdot)\|_{\dot{H}^{-1}} \geq \expig(-\mathcal{V}(heta(t,\cdot))ig)$$

► Mixing measure decay at most exponentially.

Recall main theorem:

$$\mathcal{V}(\theta(t,\cdot)) - \mathcal{V}(\theta_0) \leq C \left\|\theta_0\right\|_{\infty} \left\|\theta_0\right\|_{p'} \int_0^t \left\|\nabla u(t',\cdot)\right\|_p dt'$$

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Stronger than \dot{H}^{-1}

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Compare

$$\int_0^T \|\nabla u(t,\cdot)\|_p \ dt \leq M \implies \int |\xi|^{-2} \, |\hat{\theta}(T,\xi)|^2 \ d\xi \gtrsim \exp(-M)$$

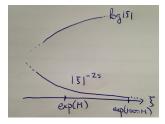
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Stronger than \dot{H}^{-1}

Compare

$$\int_0^T \left\|\nabla u(t,\cdot)\right\|_p \, dt \le M \implies \int |\xi|^{-2} \, |\hat{\theta}(T,\xi)|^2 \, d\xi \gtrsim \exp(-M)$$

and
$$\int_0^T \|\nabla u(t,\cdot)\|_p \ dt \le M \implies \int \log|\xi| \ |\hat{\theta}(T,\xi)|^2 \ d\xi \lesssim M$$



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Corollary: on the blowup of Sobolev norms

If $\|\nabla u(t,\cdot)\|_{L^2} \leq C$:

•
$$\int |\hat{\theta}(t,\xi)|^2 d\xi = C$$

•
$$\int \log |\xi| |\hat{ heta}(t,\xi)|^2 d\xi \leq C (1+t)$$

•
$$\int (\log |\xi|)^2 |\hat{ heta}(t,\xi)|^2 \, d\xi \leq C \, (1+t)^2$$

•
$$\int |\xi|^{2s} |\hat{ heta}(t,\xi)|^2 \, d\xi$$
 can blow up, for any $s>0$

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Proof

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Proof

Outline of proof

9 Write $\mathcal{V}(f) = \int \log |\xi| \, |\hat{f}(\xi)|^2 \, d\xi$ in physical space. Morally

$$\mathcal{V}(f) = \iint \frac{f(x) f(y)}{|x - y|^d} \, dx \, dy$$

2 Time derivative is

$$\frac{d}{dt}\mathcal{V}(\theta) = \iint \theta(x)\,\theta(y)\,\frac{u(x)-u(y)}{|x-y|}\cdot\frac{x-y}{|x-y|^{d+1}}\,dx\,dy$$

Incompressibility implies

$$\frac{d}{dt}\mathcal{V}(\theta) = \iint \theta(x)\,\theta(y)\,(m_{xy}\nabla u):\,\mathcal{K}(x-y)\,dx\,dy$$

O Hölder-type bounds. Mixing result follows by elementary arguments.

More details

Recall

$$\mathcal{V}(f) = \int_{\mathbf{R}^d} \log |\xi| \, |\hat{f}(\xi)|^2 \, d\xi$$

In physical space:

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$$\mathcal{V}(f) = \alpha_d \left(\frac{1}{2} \iint_{|x-y| \le 1} \frac{|f(x) - f(y)|^2}{|x-y|^d} \, dx \, dy - \iint_{|x-y| > 1} \frac{f(x)f(y)}{|x-y|^d} \, dx \, dy \right) + \beta_d \, \|f\|_{L^2}^2$$

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2 Time-derivative of \mathcal{V} along the flow $\partial_t \theta + \operatorname{div}(u\theta) = 0$:

$$\frac{d}{dt}\mathcal{V}(\theta(t,\cdot)) = c_d \operatorname{PV} \iint \theta(t,x)\theta(t,y) \frac{u(t,x) - u(t,y)}{|x-y|} \cdot \frac{x-y}{|x-y|^{d+1}} \, dx \, dy$$

Incompressibility constraint div u = 0 → multilinear singular integral can be written as

$$\frac{u(t,x)-u(t,y)}{|x-y|}\cdot\frac{x-y}{|x-y|^{d+1}}=\left(m_{xy}\nabla u(t,\cdot)\right):K(x-y)$$

with K a Calderón–Zygmund kernel.

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End of proof

Multilinear singular integral bounds (Seeger, Smart, Street):

$$\mathsf{PV} \iint \theta(t, x) \theta(t, y) (m_{xy} \nabla u(t, \cdot)) : \mathcal{K}(x - y) \, dx \, dy \le C \|\theta(t, \cdot)\|_{\infty} \|\theta(t, \cdot)\|_{p'} \|\nabla u(t, \cdot)\|_{p}$$

Hence

$$\left|\frac{d}{dt}\mathcal{V}(\theta(t,\cdot))\right| \leq C \left\|\theta_0\right\|_{\infty} \left\|\theta_0\right\|_{p'} \left\|\nabla u(t,\cdot)\right\|_{p}$$

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A remark

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A remark

A word on the harmonic analysis

Trilinear form

$$\Lambda(a,\theta,\phi) = \mathsf{PV} \iint \mathcal{K}(x-y) (m_{xy}a) \theta(x) \phi(y) \, dx \, dy$$

 $m_{xy}a = \text{average of } a \text{ on } [x, y]$

- Christ-Journé (1987): $C(a, \theta, \phi) \lesssim \|\theta\|_a \|\phi\|_{a'} \|a\|_{\infty}$
- Seeger, Smart, Street (2015): $||a||_p$ (p > 1)

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3 Summary

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Image: A matrix and a matrix

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 Summary New approach to

$$arepsilon(t) \geq C^{-1} \exp\left(-C \int_0^t \left\|
abla u(t') \right\|_p \, dt'
ight)$$

(p > 1). Use of

$$\mathcal{V}ig(heta(t,\cdot)ig) = \int \log|\xi| \, |\hat{ heta}(t,\xi)|^2 \, d\xi$$

Still open

Can we get Bressan's conjecture: unlikely. Easier version than Bressan? $L \log L$ ok. Note: no L^1 result available yet.

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Thank you for your attention!

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