A comparison principle for variational problems, with applications to optimal transport

Flavien Léger

Inria and Université Paris Dauphine

July 2, 2025

Maxime Sylvestre
Université Paris Dauphine

Comparison principles

Consider the variational problem

$$\min_{u \in X} \mathcal{E}(u, f).$$

Goal: Find structural conditions on $\mathcal{E}$ so that: ordered data $f_1 \leq f_2$ give ordered solutions

$$u(f_1) \leq u(f_2).$$

Outline

$$\mathcal{T}_c(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int c(x,y)\,d\pi = \sup_\phi \int \phi\,d\mu - \int \phi^c\,d\nu.$$

1. Comparison principle for JKO-type problems.

   Theorem. H = convex internal energy. Let $\mu_1 \leq \mu_2$ and $\nu_i = \operatorname{argmin}_\nu \mathcal{T}_c(\mu_i,\nu) + H(\nu)$. Then $\nu_1 \leq \nu_2$.

2. Comparison principle for Kantorovich potentials.

   Theorem. $\Phi_c(\mu,\nu)$ = set of Kantorovich potentials. Take $\phi_i \in \Phi_c(\mu_i,\nu)$. If $\mu_1 \leq \mu_2$ and boundary conditions, then $\phi_1 \wedge \phi_2 \in \Phi_c(\mu_1,\nu)$ and $\phi_1 \vee \phi_2 \in \Phi_c(\mu_2,\nu)$.

3. Proofs via submodularity and exchangeability.

Motivation: Why comparison principles?

$$\min_{u \in X} \mathcal{E}(u, f).$$

- Control of the solution $u(f)$: solve equation for a "simple" $f_0 \geq f$ ($f_0 =$ constant, linear, Gaussian...), then we know

$$u(f) \leq u_0 := u(f_0).$$

  Particular case: when constants are preserved we have a maximum principle: $\max u = \max f$.

- Uniqueness. u_1, u_2 two solutions for f, then $u_1 \leq u_2$ and $u_2 \leq u_1 \to u_1 = u_2$.

Motivation: Why comparison principles?

- L^1 contraction [Crandall, Tartar '80]

> Suppose that the mapping $f \mapsto u(f)$ preserves mass. Then comparison principle $f_1 \le f_2 \implies u(f_1) \le u(f_2)$ implies
> $$\|u(f_1) - u(f_2)\|_{L^1} \le \|f_1 - f_2\|_{L^1}.$$

(Exists also with an L^∞ flavor).

Comparison principle for JKO problems

Setting. Ω, Ω^* two compact metric spaces, $c \in C(\Omega \times \Omega^*)$,

$$\mathcal{T}_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int c(x, y)\, d\pi.$$

Consider the JKO problem: given $\mu \in \mathcal{M}_+(\Omega)$, solve

$$\min_{\nu \in \mathcal{M}_+(\Omega^*)} \mathcal{T}_c(\mu, \nu) + H_m(\nu).$$

Here $H_m(\nu) = \int_{\Omega^*} h(\frac{d\nu}{dm})\, dm$, where $h\colon [0, +\infty) \to \mathbb{R}$ is a strictly convex, l.s.c. and superlinear function, and $m \in \mathcal{M}_+(\Omega)$ is a fixed reference measure.

Comparison principle for JKO problems

[**L.**, Sylvestre, '25, *A comparison principle for variational problems*]

Theorem. For $i = 1, 2$, let $\mu_i \in \mathcal{M}_+(\Omega)$ and

$$\nu_i = \underset{\nu \in \mathcal{M}_+(\Omega^*)}{\operatorname{argmin}} \; \mathcal{T}_c(\mu_i, \nu) + H_m(\nu).$$

Then

$$\mu_1 \leq \mu_2 \implies \nu_1 \leq \nu_2.$$

Comparison principle for JKO problems

[**L.**, Sylvestre, '25, *A comparison principle for variational problems*]

Theorem. For $i = 1, 2$, let $\mu_i \in \mathcal{M}_+(\Omega)$ and

$$\nu_i = \underset{\nu \in \mathcal{M}_+(\Omega^*)}{\operatorname{argmin}} \ \mathcal{T}_c(\mu_i, \nu) + H_m(\nu).$$

Then

$$\mu_1 \leq \mu_2 \implies \nu_1 \leq \nu_2.$$

- Minimal assumptions. Uniqueness from assumptions on H_m.
- A similar result was obtained in [Jacobs, Kim, Tong '22] when a transport exists and c is C^1_{loc} and twisted.

Remarks

Proof via exchangeability of $\mathcal{T}_c$ allows extensions to:

- Entropic cost $\mathcal{T}_{c,\varepsilon}(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int c\, d\pi + \varepsilon\, \mathrm{KL}(\pi|R)$.
- Unbalanced cost $\mathrm{UOT}(\mu,\nu)$.
- Other nonlinearities
  $\tilde{\mathcal{T}}(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int g(x,y,d\pi/dR)\, dR$.
- $\mathrm{KL}(\mu,\nu)$ or more general Csiszár divergences $D_h(\mu,\nu)$.

Remarks

Maximum principle: if every constant is a fixed point, e.g. $c(x, y) = |x - y|^2$, then

$$\mu \leq C_0 \implies \text{solution } \nu \leq C_0.$$

Remarks

Maximum principle: if every constant is a fixed point, e.g. $c(x,y) = |x-y|^2$, then

$$\mu \leq C_0 \implies \text{solution } \nu \leq C_0.$$

Evolution: Think of

$$\mu^\tau(t+1) = \operatorname*{argmin}_\nu \frac{1}{2\tau} W_2^2(\mu^\tau(t), \nu) + H_m(\nu).$$

Then $\mu_1^\tau(0) \leq \mu_2^\tau(0)$ implies $\mu_1^\tau(t) \leq \mu_2^\tau(t)$.
As $\tau \to 0$: comparison of the continuous evolution.

Remarks

Maximum principle: if every constant is a fixed point, e.g. $c(x,y) = |x-y|^2$, then

$$\mu \leq C_0 \implies \text{solution } \nu \leq C_0.$$

Evolution: Think of

$$\mu^\tau(t+1) = \operatorname*{argmin}_\nu \frac{1}{2\tau} W_2^2(\mu^\tau(t), \nu) + H_m(\nu).$$

Then $\mu_1^\tau(0) \leq \mu_2^\tau(0)$ implies $\mu_1^\tau(t) \leq \mu_2^\tau(t)$.
As $\tau \to 0$: comparison of the continuous evolution.

L^1 **contraction**: $\|\mu_1^\tau(t) - \mu_2^\tau(t)\|_{L^1} \leq \|\mu_1^\tau(0) - \mu_2^\tau(0)\|_{L^1}$.

Comparison principle for Kantorovich potentials

Setting. Ω, Ω^* two compact metric spaces, $c \in C(\Omega \times \Omega^*)$. Then

$$\mathcal{T}_c(\mu, \nu) = \sup_{\phi \in C(\Omega)} \int_\Omega \phi \, d\mu - \int_{\Omega^*} \phi^c \, d\nu.$$

Here $\phi^c(y) = \sup_{x \in \Omega} c(x, y) - \phi(x)$.

$\Phi_c(\mu, \nu) \subset C(\Omega)$: set of solutions, $\neq \emptyset$.

Comparison principle for Kantorovich potentials

[**L.**, Sylvestre, '25, *A comparison principle for variational problems*]

Theorem. $\mu_i \in \mathcal{P}(\Omega)$, $\nu \in \mathcal{P}(\Omega^*)$, $\phi_i \in \Phi_c(\mu_i, \nu)$, $U \subset \Omega$. Then

$$\begin{cases} \mu_1 \leq \mu_2 & \text{on } U \\ \phi_1 \leq \phi_2 & \text{on } \Omega \setminus U \end{cases} \implies \begin{cases} \phi_1 \wedge \phi_2 \in \Phi_c(\mu_1, \nu) \\ \phi_1 \vee \phi_2 \in \Phi_c(\mu_2, \nu). \end{cases}$$

And $\phi_1 \leq \phi_2$ on the support of $\mu_2 - \mu_1$.

- Natural setting for principal–agent.
- Transport problem can be continuous, discrete, and can be extended to entropic OT, UOT, and so on.

Remarks

If uniqueness of Kantorovich potentials (up to an additive constant) then conclusion becomes $\phi_1 \leq \phi_2$.

If nonuniqueness, comparison principle on the solution sets.

When $\mathcal{F}_1, \mathcal{F}_2$ are sets of functions, $\mathcal{F}_1 \leq_S \mathcal{F}_2$ in the strong set order or Veinott order if

$$\forall u_1 \in \mathcal{F}_1, u_2 \in \mathcal{F}_2, \quad u_1 \wedge u_2 \in \mathcal{F}_1, \text{ and } u_1 \vee u_2 \in \mathcal{F}_2.$$

Implies in particular that $\inf \mathcal{F}_1 \leq \inf \mathcal{F}_2$ and $\sup \mathcal{F}_1 \leq \sup \mathcal{F}_2$, when inf and sup exist.

Remarks

Taking $\mu_1 = \mu_2 = \mu$: Set of Kantorovich potentials $\Phi_c(\mu, \nu)$ is stable by $\wedge$ and $\vee$ (lattice).

Recovers the comparison principle for Monge–Ampère: given a bounded open set $U \subset \mathbb{R}^n$, solve

$$\begin{cases} \det D^2 u = f \\ u \text{ is convex.} \end{cases}$$

Key insight: for any $E \subset U$,

$$\int_E \det D^2 u = (\nabla u^*)_\# \,\mathrm{Leb}\,.$$

Submodularity and exchangeability

Submodularity

Ω a compact metric space, $X = C(\Omega)$.

Definition. $E\colon X \to \mathbb{R} \cup \{+\infty\}$ is submodular if

$$E(\phi_1 \wedge \phi_2) + E(\phi_1 \vee \phi_2) \leq E(\phi_1) + E(\phi_2).$$

- Well studied in discrete optimization, combinatorics, economics.
- Naturally defined on Banach lattices $X = (X, \|\cdot\|, \leq)$.

Intuition: submodularity gives comparison principles

Consider jointly submodular $E\colon X \times Y \to \mathbb{R} \cup \{+\infty\}$.
(X, Y functional spaces).
Given data $f \in Y$, solve the variational problem

$$\min_{u \in X} E(u, f).$$

Let $f_1 \leq f_2$, with corresponding minimizer u_1, u_2. Then

$$E(u_1 \wedge u_2, f_1) + E(u_1 \vee u_2, f_2) \leq E(u_1, f_1) + E(u_2, f_2).$$

Direct consequence:

$$u_1 \wedge u_2 \in \operatorname{argmin} E(\cdot, f_1) \text{ and } u_1 \vee u_2 \in \operatorname{argmin} E(\cdot, f_2).$$

Intuition: submodularity gives comparison principles

Direct consequence:

$$u_1 \wedge u_2 \in \operatorname{argmin} E(\cdot, f_1) \text{ and } u_1 \vee u_2 \in \operatorname{argmin} E(\cdot, f_2).$$

This is an ordering of the solution sets:

$$\operatorname{argmin} E(\cdot, f_1) \leq_S \operatorname{argmin} E(\cdot, f_2).$$

Intuition: submodularity gives comparison principles

Direct consequence:

$$u_1 \wedge u_2 \in \operatorname{argmin} E(\cdot, f_1) \text{ and } u_1 \vee u_2 \in \operatorname{argmin} E(\cdot, f_2).$$

This is an ordering of the solution sets:

$$\operatorname{argmin} E(\cdot, f_1) \leq_S \operatorname{argmin} E(\cdot, f_2).$$

Suppose solution is unique.
Then $u_1 = u_1 \wedge u_2$ and $u_2 = u_1 \vee u_2$, i.e.

$$u_1 \leq u_2.$$

Submodular functions

Examples.

- $E(u) = \int h(\nabla u(x))\, dm(x)$: as particular cases, the Dirichlet energy or the perimeter
- $E(u) = \iint h(u(x) - u(y))\, dm(x, y)$ for convex h;
- $E(u) = \int g(u(x))\, dm(x)$ for arbitrary g
- $E(u, v) = -\int u(x)\, v(x)\, dm(x)$

Property: submodularity is stable by sum.

Proof of the comparison principle on Kantorovich potentials

Lemma. $K(\phi) = \int_{\Omega^*} \phi^c(y)\, d\nu(y)$ is submodular.

Proof. Let $\phi_1, \phi_2 \in C(\Omega)$ and fix $y \in \Omega^*$.

$$\phi_1(x) - c(x,y) \leq \phi_1^c(y)$$
$$\phi_2(x) - c(x,y) \leq \phi_2^c(y),$$

gives

$$(\phi_1 \wedge \phi_2)(x) - c(x,y) \leq (\phi_1^c \wedge \phi_2^c)(y),$$
$$(\phi_1 \vee \phi_2)(x) - c(x,y) \leq (\phi_1^c \vee \phi_2^c)(y).$$

Proof of the comparison principle on Kantorovich potentials

Maximizing over $x \in \Omega$:

$$(\phi_1 \wedge \phi_2)^c(y) \leq (\phi_1^c \wedge \phi_2^c)(y),$$
$$(\phi_1 \vee \phi_2)^c(y) \leq (\phi_1^c \vee \phi_2^c)(y).$$

Sum:

$$(\phi_1 \wedge \phi_2)^c(y) + (\phi_1 \vee \phi_2)^c(y) \leq \phi_1^c(y) + \phi_2^c(y).$$

Integrating over ν gives

$$K(\phi_1 \wedge \phi_2) + K(\phi_1 \vee \phi_2) \leq K(\phi_1) + K(\phi_2). \quad \square$$

Proof of the comparison principle on Kantorovich potentials

Write $\Phi_c(\mu,\nu) = \operatorname{argmin} J(\mu,\cdot)$ with $J(\mu,\phi) = K(\phi) - \int_\Omega \phi \, d\mu$.

Proof of the theorem:

$$J(\mu_1, \phi_1 \wedge \phi_2) + J(\mu_2, \phi_1 \vee \phi_2) + \int_\Omega (\phi_1 - \phi_2)^+ \, d(\mu_2 - \mu_1) \leq J(\mu_1, \phi_1) + J(\mu_2, \phi_2). \quad \square$$

Remarks:

- only relies on the submodularity of K.
- Submodularity of K is elementary.

Exchangeability

X = a Banach lattice (think $X = C(\Omega)$).

Theorem. Let $E \colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. convex function. Then E is submodular iff $F = E^*$ satisfies: for every $\mu_1, \mu_2 \in X^*$, and every $t_{21} \in [0, (\mu_2 - \mu_1)^+]$, there exists $t_{12} \in [0, (\mu_1 - \mu_2)^+]$ such that

$$F(\mu_1 + t_{21} - t_{12}) + F(\mu_2 - t_{21} + t_{12}) \leq F(\mu_1) + F(\mu_2). \quad (1)$$

Definition. $F \colon X^* \to \mathbb{R} \cup \{+\infty\}$ is exchangeable if (1) holds.

Intuition: exchangeability gives comparison principles

Given data $\eta \in Y^*$, solve for $F\colon X^* \times Y^* \to \mathbb{R} \cup \{+\infty\}$ jointly exchangeable

$$\min_{\mu \in X^*} F(\mu, \eta).$$

Take $\mu_i \in \mathcal{M}_+(\Omega)$ and $\eta_i = \operatorname{argmin} F(\mu_i, \cdot)$ unique. Then

$$\mu_1 \le \mu_2 \implies \eta_1 \ge \eta_2.$$

Proof of the comparison principle for JKO

Ideas of the proof:

- $\mathcal{T}_c(\mu,\nu) = \sup_\phi \int \phi \, d\mu - K_\nu(\phi) = K_\nu^*(\mu)$. Since K_ν is submodular, then $\mu \mapsto \mathcal{T}_c(\mu,\nu)$ is exchangeable.

- In fact $(\mu,\eta) \mapsto \mathcal{T}_c(\mu,-\eta)$ is jointly exchangeable.

- For convex internal energies H_m, the map

$$(\mu,\eta) \mapsto \mathcal{T}_c(\mu,-\eta) + H_m(-\eta)$$

is jointly exchangeable.

Thank you!

https://arxiv.org/abs/2506.18884